ON MEASURABLE STOCHASTIC PROCESSES*

BY

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In recent years probability theory has been formulated mathematically as measure theory; in the case of stochastic processes depending upon a continuous parameter the measures considered are defined on certain subspaces of the space of all functions of a real variable.† This formulation of stochastic processes depending upon a continuous parameter gives rise to certain measurability problems, and it is with these measurability problems that this paper is concerned. In particular we shall be concerned with conditions under which there will exist what Doob has called a measurable stochastic process.‡ In §1 we give the necessary mathematical formulation of the notion of a stochastic process. In §2 we obtain general conditions for the existence of a measurable process, while in §3 we use the results of §2 to obtain conditions upon the conditional probability functions which are necessary and sufficient for the existence of a measurable process. In §4 we prove a theorem which is essentially due to W. Doeblin concerning the existence of a special sort of measurable process in case the conditional probabilities satisfy certain regularity conditions.

1. **Mathematical formulations.** We shall denote by Ω the space of all real-valued functions of a real variable. We introduce a topology on Ω by defining neighborhoods as follows: if $t_1, \cdots, t_n$ is any finite set of real numbers and if $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are sets of real numbers satisfying $-\infty \leqq a_i < b_i \leqq \infty$ $(i=1, \cdots, n)$, then the set of elements $x(t)$ of Ω which satisfy $a_i < x(t_i) < b_i$ $(i=1, \cdots, n)$ is a neighborhood. Next we consider a probability measure $P(M)$, defined on the Borel field of sets determined by the collection of neighborhoods;§ we shall suppose the domain of definition of the measure $P(M)$ to be so extended that if $P(M)=0$ for a certain set M, then $P(N)$ is defined for every subset N of M. The sets for which $P(M)$ is defined will be called P-measurable. If N is any set in Ω, we define its outer P-meas-

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† See [2] and [5]. Numbers in square brackets refer to the bibliography at the end of this paper.

‡ See [2, p. 113].

§ The Borel field of sets determined by a given collection of sets is the smallest collection which contains the sets of the given collection and which has the further properties: (1) if M belongs to the collection, then so does the complement of M, (2) if $M_1, \cdots, M_n, \cdots$ belong to the collection, then so do $\sum_{i=1}^{\infty} M_i$ and $\prod_{i=1}^{\infty} M_i$. A measure is a completely additive nonnegative set-function; a probability measure is a measure for which $P(\Omega)=1$.

ure, denoted by $\overline{P(N)}$, to be the lower bound of numbers of the form $P(M)$, where M ranges through the P-measurable sets including N.

DEFINITION. *Let Ω' be a subset of Ω for which $\overline{P(\Omega')}=1$. If M' is the intersection of Ω' with a P-measurable set M, then we define the probability measure $P'(M')$ by $P'(M')=P(M)$.* The space Ω' together with the probability measure $P'(M')$ is a stochastic process.*†

For brevity we shall sometimes speak as though the space Ω' were itself the stochastic process; a reading of the context in which the term occurs will eliminate any ambiguity of meaning. We shall sometimes use the symbol ω and sometimes the symbol $x(t)$ to denote a point of Ω, using the latter symbol when we wish to emphasize the fact that a point of Ω is a function of a real variable. If ω is the function $x(t)$ and if s is a real number, then the symbol $x_s(\omega)$ will denote the value of $x(t)$ for $t=s$; this function $x_s(\omega)$ is fundamental in the measurability conditions that follow.

Let T denote the space of real numbers t; let $T\times\Omega'$ denote the product space of T with a space Ω'; then if t-measure is taken as Lebesgue measure, we can define a measure in this product space so that the measure of the direct product of a Lebesgue measurable set and a P'-measurable set is the product of their measures.‡ It is readily seen that a set in $T\times\Omega'$ is measurable if and only if it is the intersection of $T\times\Omega'$ with a set which is measurable in $T\times\Omega$.

DEFINITION. *The stochastic process consisting of Ω' with a P'-measure on it is measurable if $x_s(\omega)$ is measurable on $T\times\Omega'$.*

The measurability of this function $x_s(\omega)$ is of fundamental importance in theorems about stochastic processes. For example, it makes possible certain applications of the Birkhoff ergodic theorem.

2. **A general condition for the existence of a measurable stochastic process.** In this section we shall obtain a necessary and sufficient condition (on a P-measure) that Ω contain a measurable stochastic process. We shall then use this condition to derive an important necessary and sufficient condition, due to Kolmogoroff, for the existence of a measurable process.§

LEMMA 1. *Let x-space be any space on which a measure is defined, and let E be any set lying in x-space. Suppose that $f(x)$ is a real-valued function, defined*

* It is shown in [2] that the value of the function $P'(M')$ is independent of the particular choice of the set M used in representing M' as the intersection of Ω' and a P-measurable set.

† This definition (as well as the definition of a measurable stochastic process which follows) is that given in [2]; see pp. 109–110 and p. 113. Also we remark that this is not the most general definition of a stochastic process but only a definition of the type of stochastic process to be studied in this paper.

‡ For a discussion of measures in product spaces see [6].

on E, with the property that for every real number k, the x-set (lying in E) for which $f(x) > k$ is the intersection of E with a measurable x-set. Then $f(x)$ can be so defined for x in the complement of E that it will become a measurable x-function.

Proof. By hypothesis we know that for every rational number r there is a measurable x-set $N(r)$, such that

$$_x[f(x) > r] = N(r) \cdot E.^*$$

We now define the measurable x-sets $M(r)$ by $M(r) = \sum_{r' \geqq r} N(r')$, and it is obvious that

$$_x[f(x) > r] = M(r) \cdot E.$$

We define $f(x)$ for x lying in the complement of E as follows:

$$\begin{aligned} f(x) &= \text{u.b.}_{x \varepsilon M(r)} r \quad \text{for} \quad x \varepsilon \sum_r M(r) \\ &= 0 \qquad\qquad\quad \text{for} \quad x \notin \sum_r M(r). \end{aligned}$$

It is readily verified that $f(x)$ becomes a measurable x-function when so extended.

THEOREM 1. *The space Ω contains a measurable stochastic process if and only if P-measure is so defined that there is a measurable function on $T \times \Omega$ which for each fixed t equals $x_t(\omega)$ at almost all ω-points.*

Proof. We shall first show that if there is a measurable stochastic process, then there is a function $f(t, \omega)$ of the sort mentioned in this theorem. Suppose that Ω' is a measurable stochastic process; since the function $x_t(\omega)$ is measurable on $T \times \Omega'$ and since this means that for every real number k the set in $T \times \Omega'$ for which $x_t(\omega) > k$ is the intersection of $T \times \Omega'$ with a measurable set in $T \times \Omega$, we can apply Lemma 1 to this function (taking the x-space mentioned in that lemma to be $T \times \Omega$ and taking $T \times \Omega'$ for the set E). This lemma tells us that there is a measurable function on $T \times \Omega$, $g(t, \omega)$, which equals $x_t(\omega)$ on $T \times \Omega'$. Now for fixed t the function $g(t, \omega)$ is a measurable ω-function except possibly for a t-set whose measure is zero. We define the function $f(t, \omega)$ to be equal to $g(t, \omega)$, except for t in this exceptional set; for these exceptional t-values we define $f(t, \omega)$ to be equal to $x_t(\omega)$. Then the function $f(t, \omega)$ is measurable on $T \times \Omega$ and is also P-measurable for each value of t. We see that for each t the functions $f(t, \omega)$ and $x_t(\omega)$ are equal at almost all ω-points since they are both P-measurable and they are equal on Ω'.

* The symbol $_x[f(x) > r]$ denotes the set of x-points for which $f(x) > r$.

Next we shall show that if there is a measurable function on $T\times\Omega$, as mentioned in the theorem, then Ω contains a measurable stochastic process. Doob has shown* that if $f(t, \omega)$ is a measurable (t, ω)-function and if $\varphi_n(t)$ is defined by $\varphi_n(t) = k2^{-n}$ for $(k-1)2^{-n} < t \leqq k2^{-n}$, then there is a real number c and an increasing sequence of integers $\{a_n\}$ such that

$$\lim_{n\to\infty} f(\varphi_{a_n}(t) + c, \omega) = f(t + c, \omega)$$

at almost all (t, ω)-points. This implies that for almost all t

$$P\left(\lim_{n\to\infty} f(\varphi_{a_n}(t) + c, \omega) = f(t + c, \omega)\right) = 1$$

and hence (since the functions $f(t, \omega)$ and $x_t(\omega)$ are equal, for fixed t, at almost all ω-points) that for almost all t

$$P\left(\lim_{n\to\infty} x(\varphi_{a_n}(t) + c) = x(c + t)\right) = 1,$$

and hence, according to a theorem of Doob,† Ω contains a measurable stochastic process.

The next theorem which we shall prove enables us to obtain from Theorem 1 a condition found by Kolmogoroff to be necessary and sufficient for the existence of a measurable stochastic process; first, however, we introduce a few preliminary notions.

If $f(\omega)$ and $g(\omega)$ are P-measurable functions, we define the distance between them to be

$$\int_\Omega |\arctan f(\omega) - \arctan g(\omega)|\, dP.$$

Now let $f(t, \omega)$ be a (t, ω)-function which is P-measurable for each t. We consider the related function $\Phi(t)$ whose value for each t is the ω-function $f(t, \omega)$. Using the above definition of distance in the space of P-measurable functions, we see that the range of $\Phi(t)$ is a metric space. We shall say that $\Phi(t)$ is measurable if the origin of any sphere in its range space is Lebesgue measurable.

THEOREM 2. *$\Phi(t)$ is approximately continuous almost everywhere if and only if there is a measurable function on $T\times\Omega$ which for each t equals $f(t, \omega)$ at almost all ω-points.*‡

Proof. Suppose first that $\Phi(t)$ is approximately continuous almost every-

* See [2, p. 113].
† See [2, p. 115].
‡ This theorem is a generalization of a theorem due to Doob; see [4, p. 757].

where; from a lemma of Doob* it follows that there is a t-set T_0 of measure zero such that the range of $\Phi(t)$ for $t \notin T_0$ is separable. This means that there is a sequence of real numbers, $\{\alpha_n\}$, such that the sequence of ω-functions $\{f(\alpha_n, \omega)\}$ is dense in the family of ω-functions $\{f(t, \omega)\}$, $t \notin T_0$. We can find for each function of this sequence a separable collection† of P-measurable sets with respect to which it is measurable. Hence it follows that there is a single separable collection of P-measurable sets with respect to which every function of the sequence $\{f(\alpha_n, \omega)\}$ is measurable. Then, since every member of the family $\{f(t, \omega)\}$, $t \notin T_0$, can be approximated by members of this sequence, it follows that every member of the family $\{f(t, \omega)\}$, $t \notin T_0$, is measurable with respect to this separable collection of sets.

Because we are assuming $\Phi(t)$ to be approximately continuous almost everywhere, it follows immediately that for any P-measurable set M, the t-function $\int_M \text{arc tan}\, f(t,\omega)dP$ is approximately continuous almost everywhere and hence is measurable. This fact and the fact established in the preceding paragraph together imply, by Theorem 1 of [4], that there is a measurable function on $T \times \Omega$ which equals $f(t, \omega)$, for each fixed t, at almost all ω-points.

We shall now show that if there is a measurable function on $T \times \Omega$ which is so related to $f(t, \omega)$, then $\Phi(t)$ is approximately continuous almost everywhere. We shall show this by showing that $\Phi(t)$ is measurable and that there is a t-set T_0 of measure zero such that the range of $\Phi(t)$ for $t \notin T_0$ is separable. To show that $\Phi(t)$ is measurable we must show that for any sphere S in the space of P-measurable functions the t-set for which $\Phi(t)$ belongs to S is measurable; that is, we must show that if $g(\omega)$ is any P-measurable function and k is any real number, then the t-set for which

$$\int \left| \text{arc tan}\, f(t, \omega) - \text{arc tan}\, g(\omega) \right| dP < k$$

is measurable. If $f_0(t, \omega)$ is the function which is measurable on $T \times \Omega$ and equals $f(t, \omega)$, for each fixed t, at almost all ω-points, then the function $\left|\text{arc tan}\, f_0(t, \omega) - \text{arc tan}\, g(\omega)\right|$ will be a measurable (t, ω)-function so that, by Fubini's theorem,

$$\int \left| \text{arc tan}\, f(t, \omega) - \text{arc tan}\, g(\omega) \right| dP \equiv \int \left| \text{arc tan}\, f_0(t, \omega) - \text{arc tan}\, g(\omega) \right| dP$$

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* See [4, p. 755, Lemma 1]. The portion of the lemma which we use can be stated (where $\Phi(t)$ still represents the ω-function $f(t, \omega)$): $\Phi(t)$ is approximately continuous almost everywhere if and only if $\Phi(t)$ is measurable and there is a t-set T_0 of measure zero such that the range of $\Phi(t)$, for $t \notin T_0$, is a separable space.

† A collection of P-measurable sets $\{M_\alpha\}$ with characteristic function $\{\psi_\alpha(\omega)\}$ is said to be separable if the ω-functions $\{\psi_\alpha(\omega)\}$ form a separable space with the metric defined above.

is a measurable t-function; hence the t-set defined by the above inequality is measurable.

By Theorem 1 of [4] we see that there is a t-set T_0 of measure zero and a separable collection F of P-measurable sets such that for $t \notin T_0$, $f(t, \omega)$ is measurable with respect to F. This means that there exists a sequence of P-measurable sets $\{M_n\}$ with characteristic functions $\{\psi_n(\omega)\}$, such that if N is any set belonging to F, with characteristic function $\psi(\omega)$, then there is a subsequence $\{\psi_{\alpha_n}(\omega)\}$ of $\{\psi_n(\omega)\}$ such that

$$\int \left| \text{arc tan } \psi_{\alpha_n}(\omega) - \text{arc tan } \psi(\omega) \right| dP \to 0, \qquad n \to \infty .$$

It is readily seen that the collection of linear combinations, with rational coefficients, of these functions $\{\psi_n(\omega)\}$ is dense in the family $\{f(t, \omega)\}$, $t \notin T_0$, and hence that the range of $\Phi(t)$, $t \notin T_0$, is separable.

We are now able to obtain the Kolmogoroff theorem for the existence of a measurable process:

THEOREM 3. *Ω contains a measurable stochastic process if and only if P-measure is so defined that for every $\epsilon > 0$ and almost all t*

$$P(\left| x(t + h) - x(t) \right| > \epsilon) \to 0$$

as h approaches zero on an h-set, which may depend upon t but not upon ϵ, having density 1 *at $h = 0$.**

Before proving this theorem we restate it as follows: Ω contains a measurable stochastic process if and only if P-measure is so defined that $x(t)$ is approximately continuous in P-measure at almost all t-points.

Let $\Phi(t)$ be the function whose value for each t is the ω-function $x_t(\omega)$. Then from Theorems 1 and 2 we know that there is a measurable process if and only if $\Phi(t)$ is approximately continuous almost everywhere, that is, if and only if for almost every t there is an h-set of density 1 at $h = 0$ such that

$$\int \left| \text{arc tan } x_{t+h}(\omega) - \text{arc tan } x_t(\omega) \right| dP \to 0$$

as h approaches zero on that h-set. This condition is equivalent to the condition that for every $\epsilon > 0$

$$P(\left| \text{arc tan } x_{t+h}(\omega) - \text{arc tan } x_t(\omega) \right| > \epsilon) \to 0$$

and hence to the condition that for every $\epsilon > 0$

* Kolmogoroff has not yet published his proof of this theorem; it is stated in a footnote on page 757 of [4].

$$P(|x(t+h)-x(t)|>\epsilon)\to 0$$

so that the approximate continuity almost everywhere of $\Phi(t)$ is equivalent to this condition of Kolmogoroff; hence this condition is necessary and sufficient for the existence of a measurable process.

3. **Measurable stochastic processes and the conditional probability functions.** In this section we shall define the conditional probability functions of a stochastic process. We shall then obtain necessary and sufficient conditions upon them that Ω contain a measurable stochastic process. The definition which we use of these functions is due to Kolmogoroff [5, pp. 41–43].*

Let M be a P-measurable set and let $t_1, \cdots, t_n$ be a set of real numbers. Let N be a P-measurable set depending only upon $x_{t_1}, \cdots, x_{t_n}$† and consider the set-function $P(MN)$. As N ranges over the P-measurable sets which depend only upon $x_{t_1}, \cdots, x_{t_n}$, this is a completely additive nonnegative set-function which vanishes whenever $P(N)=0$. Hence there is a nonnegative P-measurable function which depends only upon $x_{t_1}, \cdots, x_{t_n}$ and which we shall denote by

$$P(x_{t_1}(\omega), \cdots, x_{t_n}(\omega); M)$$

such that

$$P(M\cdot N)=\int_N P(x_{t_1}(\omega), \cdots, x_{t_n}(\omega); M)dP.\ddagger$$

A discussion of the properties of these conditional probabilities is to be found in [5].

Before stating our next theorem we shall establish the following notation. If E is a set of real numbers, then we shall denote by E_t the ω-set, ${}_\omega[x_t(\omega)\in E]$. If I is an open interval of real numbers, then the set of inner points of the complement of I is called the *open complement* of I and is denoted by J.

THEOREM 4. *If t is a real number and if E is a set (of real numbers) with zero as a limit point, then the following assertions are equivalent:*

(1) *For every $\epsilon>0$, $P(|x(t+h)-x(t)|>\epsilon)\to 0$ $(h\to 0, h\in E)$.*

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* In this connection, also see [2, 3].

† To say that N depends only upon $x_{t_1}, \cdots, x_{t_n}$ means that if $x(t)$ and $x'(t)$ are any two points of Ω and if $x(t_i)=x'(t_i)$ $(i=1, \cdots, n)$, then either both or neither of them belongs to N. To say that an ω-function depends only upon $x_{t_1}, \cdots, x_{t_n}$ means similarly that if $x(t)$ and $x'(t)$ are any two elements of Ω and if $x(t_i)=x'(t_i)$ $(i=1, \cdots, n)$, then the function has the same value at these two points.

‡ The existence of such a function is a consequence of the Radon-Nikodym theorem. (See [8, p. 36].) That the function depends (in this case) only upon $x_{t_1}, \cdots, x_{t_n}$ is readily verified. This function is of course uniquely defined only up to a set of measure zero.

(2) *For every open interval* (*of real numbers*) *I with open complement J*

$$P(I_t \cdot J_{t+h}) + P(J_t \cdot I_{t+h}) \to 0,^* \qquad h \to 0, \ h \in E.$$

Proof. We shall show first that (1) implies (2). Let I be any open interval with open complement J and let $\{I^n\}$ be the sequence of intervals defined by I^n = points at distance greater than $1/n$ from J so that the intervals I^n expand to I as $n \to \infty$. Then for $\omega \in I_t^n \cdot J_{t+h}$ we have $|x_t(\omega) - x_{t+h}(\omega)| > 1/n$ so that (1) implies, for fixed n,

$$\text{(a)} \qquad P(I_t^n \cdot J_{t+h}) \to 0, \qquad h \to 0, \ h \in E.$$

Because the sets I^n expand to I we have, uniformly in h,

$$\text{(b)} \qquad P(I_t^n \cdot J_{t+h}) \to P(I_t \cdot J_{t+h}).$$

Then (a) and (b) together imply that

$$P(I_t \cdot J_{t+h}) \to 0, \qquad h \to 0, \ h \in E.$$

In the same way it can be shown that $P(J_t \cdot I_{t+h}) \to 0$, $h \to 0$, $h \in E$, so that (2) must hold.

To show that (2) implies (1) we shall show that (with $\epsilon > 0$ arbitrarily preassigned) for any $\delta > 0$ there is an $\eta > 0$ such that

$$P(|x(t+h) - x(t)| > \epsilon) < \delta \quad \text{for} \quad |h| < \eta, \qquad h \in E.$$

Given $\delta > 0$, first choose M so large that $P(|x(t)| \geqq M) < \delta/2$. Then cover the interval $(-M, M)$ with a finite number of open intervals each of length less than ϵ. If p is the number of intervals in this covering, then we can find a positive number η so small that for every interval I of the covering, with open complement J, we have

$$P(I_t \cdot J_{t+h}) < \delta/2p, \quad \text{for} \quad |h| < \eta, \qquad h \in E,$$

and because of

$${}_\omega[|x(t+h) - x(t)| > \epsilon] \subset \sum I_t \cdot J_{t+h} + {}_\omega[|x(t)| \geqq M]$$

we have

$$P(|x(t+h) - x(t)| > \epsilon) < \sum \delta/2p + \delta/2 = \delta \quad \text{for} \quad |h| < \eta, \qquad h \in E.$$

For our next theorem we shall have to consider for each interval I, with open complement J, the set $K = J + I$. We shall call K the set *corresponding to* I. With this understood we can state

* In case the end points a and b of I are such that the two ω-sets ${}_\omega[x_t(\omega) = a]$ and ${}_\omega[x_t(\omega) = b]$ both have P-measure zero, this condition can be replaced by the condition $P(I_t \cdot CI_{t+h}) + P(I_{t+h} \cdot CI_t) \to 0$.

THEOREM 5. *If t is a real number and E is a set (of real numbers) with zero as a limit point, then the following assertions are equivalent:*

(1) *For every open interval (of real numbers) I with open complement J,*

$$P(I_t \cdot J_{t+h}) + P(J_t \cdot I_{t+h}) \to 0, \qquad h \to 0,\ h \in E.$$

(2) *For every open interval I and corresponding set K, we have*

$$P(K_t \cdot {}_\omega[\,|\,P(x_t(\omega); I_{t+h}) - \varphi_I(x_t)\,|\, > \epsilon]) \to 0, \qquad h \to 0,\ h \in E,$$

*for every $\epsilon > 0$.**

Proof. To see that (1) implies (2) we note that $P(J_t \cdot I_{t+h}) \to 0$ says that

$$\int_{J_t} P(x_t(\omega); I_{t+h}) dP \to 0;$$

and since this integrand is nonnegative, this implies that

$$\text{(a)} \qquad P(J_t \cdot {}_\omega[P(x_t(\omega); I_{t+h}) > \epsilon]) \to 0, \qquad h \to 0,\ h \in E.$$

From the fact that $P(I_t \cdot J_{t+h}) \to 0$ ($h \to 0$, $h \in E$) we can show similarly that

$$\text{(b)} \qquad P(I_t \cdot {}_\omega[\,|\,P(x_t(\omega); I_{t+h}) - 1\,|\, > \epsilon]) \to 0, \qquad h \to 0,\ h \in E.$$

Statements (a) and (b) together imply (2).

(2) says that $P(x_t(\omega); I_{t+h})$ converges, on K_t, in P-measure to $\varphi_I(x_t)$ as h approaches zero on E; hence

$$\int_{J_t} P(x_t(\omega); I_{t+h}) dP \to \int_{J_t} \varphi_I(x_t) dP, \qquad h \to 0,\ h \in E,$$

that is, $P(J_t \cdot I_{t+h}) \to 0$, $h \to 0$, $h \in E$. It follows similarly that $P(I_t \cdot J_{t+h}) \to 0$ and hence (2) implies (1).

Combining Theorems 3, 4, and 5 we can state

THEOREM 6. *Each of the following conditions is necessary and sufficient for Ω to contain a measurable stochastic process:*

(1) *For almost every t there is an h-set of density 1 at $h = 0$ such that for every open interval of real numbers I with open complement J*

$$P(I_t \cdot J_{t+h}) + P(J_t \cdot I_{t+h}) \to 0$$

as h approaches zero on that h-set.

(2) *For almost every t there is an h-set of density 1 at $h = 0$ such that for every*

* In case the end points a and b of I are such that the ω-sets ${}_\omega[x_t(\omega) = a]$ and ${}_\omega[x_t(\omega) = b]$ both have P-measure zero, this condition can be replaced by the condition $P(|P(x_t(\omega); I_{t+h}) - \varphi_I(x_t)| > \epsilon) \to 0$. The symbol $\varphi_I(x_t)$ denotes the characteristic function of the ω-set I_t.

open interval of real numbers I, and corresponding set K, we have for every $\epsilon>0$

$$P(K_t\cdot{}_\omega[\,|\,P(x_t(\omega);\,I_{t+h})\,-\,\varphi_I(x_t)\,|\,>\,\epsilon])\to 0$$

as h approaches zero on that h-set.

DEFINITION. *Suppose that for every open interval I of real numbers there is a function* $\varphi(t,h)$ *with the property that for almost all t there is an h-set of density* 1 *at* $h=0$ *such that* $\varphi(t,h)\to 0$ *as h approaches zero on that h-set, and such that for every real number s*

$$\int |\,P(x_s(\omega);\,I_{t+h})\,-\,P(x_s(\omega);\,I_t)\,|\,dP\,<\,\varphi(t,h).$$

Then we shall say that the conditional probabilities are uniformly dominated.

THEOREM 7. *Suppose P-measure so defined that for every pair of real numbers a and t the ω-set for which* $x_t(\omega)=a$ *has P-measure zero. Then Ω contains a measurable stochastic process if and only if the conditional probabilities are uniformly dominated.*

Proof. We shall prove this theorem by showing that under the hypothesis of this theorem the conditional probabilities are uniformly dominated if and only if condition (2) of Theorem 6 is satisfied. Under the hypothesis made here the open complement of I may be replaced in condition (2) of Theorem 6 by the complement of I. Suppose now that the conditional probabilities are uniformly dominated; then

$$\begin{aligned}P(CI_t\cdot I_{t+h}) &= P(CI_t\cdot I_{t+h})\,-\,P(CI_t\cdot I_t)\\&=\int_{CI_t}\{P(x_t(\omega);\,I_{t+h})\,-\,P(x_t(\omega);\,I_t)\}\,dP\\&\leqq\int_{\Omega}|\,P(x_t(\omega);\,I_{t+h})\,-\,P(x_t(\omega);\,I_t)\,|\,dP\\&<\varphi(t,h).\end{aligned}$$

It can be shown similarly that $P(I_{t+h}\cdot CI_t)<\varphi(t,h)$. Hence condition (2) of Theorem 6 is satisfied.

Because we have, for any P-measurable set M,

$$|\,P(M\cdot I_t)\,-\,P(M\cdot I_{t+h})\,|\,\leqq\,P(I_t\cdot CI_{t+h})\,+\,P(I_{t+h}\cdot CI_t),$$

we have, for any P-measurable set F_s, depending only upon x_s

$$\left|\int_{F_s}P(x_s(\omega);\,I_t)dP\,-\int_{F_s}P(x_s(\omega);\,I_{t+h})dP\right|\,\leqq\,P(I_t\cdot CI_{t+h})\,+\,P(I_{t+h}\cdot CI_t).$$

Since this last inequality is independent of the set F_s, we can conclude that

$$\int \left| P(x_s(\omega); I_t) - P(x_s(\omega); I_{t+h}) \right| dP \leqq 2\{P(I_t \cdot CI_{t+h}) + P(I_{t+h} \cdot CI_t)\},$$

and from this inequality we see that if condition (2) of Theorem 6 holds, then the conditional probabilities must be uniformly dominated.

4. **A theorem of W. Doeblin.** We have been concerned so far throughout this paper with the existence of a measurable process, but we have not inquired into any of the properties of that process. In this section we shall prove a theorem, due essentially to W. Doeblin, concerning the existence of a measurable process whose elements satisfy a certain regularity condition.

We shall say that Ω is a Markoff process if whenever $t_1 < t_2 < \cdots < t_n$ are real numbers and M is a P-measurable set depending only upon $\{x_\alpha\}$, with all the α's greater than or equal to t_n, then

$$P(x_{t_1}(\omega), \cdots, x_{t_n}(\omega); M) = P(x_{t_n}(\omega); M).$$

If it is possible to define the conditional probability functions so that* for every $\epsilon > 0$ we have

$$P(x_t(\omega); \left| x_t(\omega) - x_{t+h}(\omega) \right| < \epsilon) \to 1$$

uniformly in ω and t, then we shall say that the conditional probabilities satisfy the *Doeblin condition*. In these terms we state the theorem of Doeblin:

THEOREM 8. *If Ω is a Markoff process whose conditional probabilities satisfy the Doeblin condition, then there is a measurable stochastic process whose elements $x(t)$ are continuous almost everywhere on the t axis, with probability* 1, *and $x(t)$ is continuous at $t = t_0$ with probability* 1, *for every value of t_0.*†

Proof. Because of Theorem 2.5 (i) of [2] it is sufficient to show that $t_n \to t$ implies $P(\lim_{n\to\infty} x(t_n) = x(t)) = 1$.‡ We shall show separately that $t_n \downarrow t$ implies $P(x(t_n) \to x(t)_{n\to\infty}) = 1$ and that $t_n \uparrow t$ implies $P(x(t_n) \to x(t)_{n\to\infty}) = 1$. From these facts it follows readily that $t_n \to t$ implies $P(x(t_n) \to x(t)_{n\to\infty}) = 1$.

First suppose $t_n \downarrow t$. Then define the ω-sets

$$M_n = {}_\omega[\text{for some } k \geqq n\text{: } \left| x(t_k) - x(t) \right| > \epsilon],$$
$$M_{nj} = {}_\omega[\text{for some } k \text{ among } n, n+1, \cdots, n+j\text{: } \left| x(t_k) - x(t) \right| > \epsilon],$$
$$M_{nji} = {}_\omega[\left| x(t_{n+i}) - x(t) \right| > \epsilon] \cdot \prod_{j \leqq l \leqq i+1} {}_\omega[\left| x(t_{n+l}) - x(t) \right| \leqq \epsilon].$$

* Because the conditional probabilities are only defined uniquely up to a set of measure zero, there will be, in general, infinitely many choices for them; if for any one of those choices this condition is fulfilled, then the conditional probabilities will be said to satisfy the Doeblin condition.

† See [1, pp. 49–53].

‡ Because we shall show this for *every* value of t it will follow (from the proof of Theorem 2.5 (i) of [2] that $x(t)$ is continuous at $t = t_0$ with probability 1, for *every* value of t_0.

We observe that $M_{nj}=\sum_{i=0}^{j}M_{nji}$, $M_{nji}\cdot M_{njk}=0$ for $i\neq k$, $M_{nj}\subset M_{n,j+1}$, $M_n =\sum_{j=0}^{\infty}M_{nj}$. It is easily seen that

$$_{\omega}[\,|\,x(t_n)-x(t)\,|>\epsilon/2]\supset\sum_{i=0}^{j}M_{nji}\cdot{}_{\omega}[\,|\,x(t_{n+i})-x(t_n)\,|<\epsilon/2]$$

and hence that

$$(\alpha)\qquad \begin{aligned} P(\,|\,x(t_n)-x(t)\,|>\epsilon/2) &\geqq P\left(\sum_{i=0}^{j}M_{nji}\cdot{}_{\omega}[\,|\,x(t_{n+i})-x(t_n)\,|<\epsilon/2]\right)\\ &=\sum_{i=0}^{j}P(M_{nji}\cdot{}_{\omega}[\,|\,x(t_{n+i})-x(t_n)\,|<\epsilon/2]).\end{aligned}$$

Now because M_{nji} depends only upon x_t, $x_{t_{n+j}}, \cdots, x_{t_{n+i}}$ and because the process is a Markoff process, we have

$$\begin{aligned} &P(M_{nji}\cdot{}_{\omega}[\,|\,x(t_{n+i})-x(t_n)\,|<\epsilon/2])\\ &\qquad=\int_{M_{nji}}P(x_{t_{n+i}}(\omega);\ |\,x(t_{n+i})-x(t_n)\,|<\epsilon/2)dP.\end{aligned}$$

If we choose n sufficiently large, the function

$$P(x_{t_{n+i}}(\omega);\ |\,x(t_{n+i})-x(t_n)\,|<\epsilon/2)$$

will be greater than 1/2 uniformly in ω and i, since the conditional probabilities satisfy the Doeblin condition. Hence for n sufficiently large we have, unifomly in i,

$$P(M_{nji}\cdot{}_{\omega}[\,|\,x(t_{n+i})-x(t_n)\,|<\epsilon/2])>(1/2)P(M_{nji}),$$

and using this fact in the inequality (α) we have

$$P(\,|\,x(t_n)-x(t)\,|>\epsilon/2)\geqq\sum_{i=0}^{j}(1/2)P(M_{nji})=(1/2)P(M_{nj}).$$

Hence

$$P(M_{nj})\leqq 2P(\,|\,x(t_n)-x(t)\,|>\epsilon/2)$$

for n sufficiently large, uniformly in j. Since the sets M_{nj} expand (as $j\to\infty$) to M_n, we have

$$(\beta)\qquad P(M_n)\leqq 2P(\,|\,x(t_n)-x(t)\,|>\epsilon/2).$$

Because the Doeblin condition implies that for every open interval I with corresponding set K

$$P(K_t\cdot{}_{\omega}[\,|\,P(x_t(\omega);I_{t_n})-\varphi_I(x_t)\,|>\epsilon])\to 0\qquad(n\to\infty),$$

we see by Theorems 4 and 5 that $P(|x(t_n)-x(t)|>\epsilon/2)\to 0$ $(n\to\infty)$ so that (β) implies $P(M_n)\to 0$ $(n\to\infty)$. This implies that $P(\lim_{n\to\infty} x(t_n)=x(t))=1$.

We shall now show that $t_n\uparrow t$ implies $P(x(t_n)\to x(t)_{n\to\infty})=1$. For each positive integer n we define the sequence of sets

$$A_n^{n+1} = {}_\omega[\,|\,x(t_{n+1}) - x(t_n)\,| > \epsilon\,],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$A_n^{n+j} = {}_\omega[\,|\,x(t_{n+j}) - x(t_n)\,| > \epsilon\,]\cdot \prod_{i=1}^{j-1} {}_\omega[\,|\,x(t_{n+i}) - x(t_n)\,| \leqq \epsilon\,],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$

and we define the set A_n by

$$A_n = {}_\omega[\text{for some } k > n\text{: } |\,x(t_k) - x(t_n)\,| > \epsilon\,].$$

We note that A_n^{n+i} depends only upon $x_{t_n}, \cdots, x_{t_{n+j}}$. Also we have $A_n = \sum_{j=1}^{\infty} A_n^{n+j}$. It is easily seen that

$$_\omega[\,|\,x(t_n) - x(t)\,| > \epsilon/2\,] \supset \sum_{j=1}^{\infty} A_n^{n+j} \cdot {}_\omega[\,|\,x(t_{n+j}) - x(t)\,| < \epsilon/2\,]$$

and hence that

$$(\gamma)\qquad \begin{aligned} P(\,|\,x(t_n) - x(t)\,| > \epsilon/2) &\geqq P\left(\sum_{j=1}^{\infty} A_n^{n+j} \cdot {}_\omega[\,|\,x(t_{n+j}) - x(t)\,| < \epsilon/2\,]\right) \\ &= \sum_{j=1}^{\infty} P(A_n^{n+j} \cdot {}_\omega[\,|\,x(t_{n+j}) - x(t)\,| < \epsilon/2\,]). \end{aligned}$$

Now because A_n^{n+i} depends only upon $x_{t_n}, \cdots, x_{t_{n+j}}$ we have

$$\begin{aligned} &P(A_n^{n+j} \cdot {}_\omega[\,|\,x(t_{n+j}) - x(t)\,| < \epsilon/2\,]) \\ &\qquad = \int_{A_n^{n+j}} P(x_{t_{n+j}}(\omega);\ |\,x(t_{n+j}) - x(t)\,| < \epsilon/2)dP. \end{aligned}$$

As above we see that for n sufficiently large this integrand is greater than 1/2, uniformly in j, and hence for n sufficiently large

$$P(A_n^{n+j} \cdot {}_\omega[\,|\,x(t_{n+j}) - x(t)\,| < \epsilon/2\,]) > (1/2)P(A_n^{n+j}).$$

This implies when used in (γ) that

$$P(\,|\,x(t_n) - x(t)\,| > \epsilon/2) \geqq \sum_{j=1}^{\infty} (1/2)P(A_n^{n+j}) = (1/2)P(A_n)$$

for n sufficiently large. The Doeblin condition implies, by virtue of Theorems 4 and 5, that the expression on the left tends to zero as $n\to\infty$ and hence that $P(A_n)\to 0$ as $n\to\infty$. This means that the sequence $\{x(t_n)\}$ satisfies the Cauchy condition, with probability 1, and hence converges, with probability 1, as $n\to\infty$. That it converges to $x(t)$ with probability 1 then follows from the fact that the left side of the above expression tends to 0 as n becomes infinite.

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